2-How to find solutions for a System

Ex1. Let $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$, compute the eigenvalues of A, find a set of eigenvectors and find a fundamental matrix for $\vec{Y}' = A\vec{Y}$.

* Find eigenvalues of A: $P(\lambda) = \det\begin{bmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{bmatrix} = 0 \iff P(\lambda) = (5-\lambda)(1-\lambda)+3=0$

or $\lambda^2 - 6\lambda + 5 + 3 = 0 \iff \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2) = 0 \rightarrow \lambda_1 = 4, \ \lambda_2 = 2$

* For each eigenvalue find the associated eigenvector

$\boxed{\text{For } \lambda_1 = 4}$ Solve $\begin{bmatrix} 5-4 & -1 \\ 3 & 1-4 \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\iff \left.\begin{matrix} v_1 - v_2 = 0 \\ 3v_1 - 3v_2 = 0 \end{matrix}\right\} \iff v_1 = v_2$ the associated eigenvector is of the form $v_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the associated solution is $\phi_1(t) = e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\boxed{\text{For } \lambda_2 = 2}$ solve $\begin{bmatrix} 5-2 & -1 \\ 3 & 1-2 \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

then $3v_1 - v_2 = 0 \iff 3v_1 = v_2$, the associated eigenvector is of the form $v_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and the corresponding solution is $\phi_2(t) = e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\therefore$ A fundamental matrix for the system is

$$\boxed{\Phi(t) = \left[e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \right]}$$

and the general solution in Matrix form is $\boxed{\vec{Y}(t) = \Phi(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$

Matrix form is $\boxed{\vec{Y}(t) = \Phi(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$

Ex2. Let $A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$, compute the eigenvalues of A, find a set of eigenvectors and find a fundamental matrix for $\vec{Y}' = A\vec{Y}$.

* to find eigenvalues we solve $P(\lambda) = \det\begin{bmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{bmatrix} = 0$ or

$$P(\lambda) = (3-\lambda)(-2-\lambda) + 4 = \lambda^2 - \lambda - 6 + 4 = 0 \iff P(\lambda) = \lambda^2 - \lambda - 2 = 0$$

$$P(\lambda) = (\lambda - 2)(\lambda + 1) = 0 \quad \begin{array}{l} \nearrow \lambda_1 = 2 \\ \searrow \lambda_2 = -1 \end{array}$$

* For each eigenvalue we find the associated eigenvector

$\boxed{\text{For } \lambda_1 = 2}$ Solve $\begin{pmatrix} 3-2 & -1 \\ 4 & -2-2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or

$\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff v_1 - v_2 = 0 \iff v_1 = v_2$ the corresponding eigenvector is of the form $v_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the solution associated with the eigenpair $\left(2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$ is $\phi_1(t) = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\boxed{\text{For } \lambda_2 = -1}$ we solve $\begin{pmatrix} 3+1 & -1 \\ 4 & -2+1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff$

$$\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff 4v_1 - v_2 = 0 \iff 4v_1 = v_2$$ then the eigenvector is of the form $v_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and the solution associated with eigenpair $\left(-1, \begin{pmatrix} 1 \\ 4 \end{pmatrix}\right)$ is $\phi_2(t) = e^{-t}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\therefore$ the F. Matrix for the system is $\boxed{\Phi(t) = \begin{bmatrix} e^{2t} & e^{-t} \\ e^{2t} & 4e^{-t} \end{bmatrix}}$ and the General solution is $\boxed{\vec{Y}(t) = \Phi(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$

Ex3. Let $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, compute the eigenvalues of A, find a set of eigenvectors and find a fundamental matrix for $\vec{Y}' = A\vec{Y}$.

(*) Eigenvalues: Solve $P(\lambda) = \det\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} = 0 \iff$

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$P(\lambda) = (2+\lambda)^2 - 1 = 0 \iff \lambda^2 + 4\lambda + 4 - 1 = 0 \iff \lambda^2 + 4\lambda + 3 = 0$

or $(\lambda+1)(\lambda+3) = 0$ $\left.\begin{matrix} \lambda_1 = -3 \\ \lambda_2 = -1 \end{matrix}\right\}$ eigenvalues of A

(*) Eigenvectors:

$\boxed{\text{For } \lambda_1 = -3}$ Solve $\begin{bmatrix} -2+3 & 1 \\ 1 & -2+3 \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0}$

or $v_1 + v_2 = 0 \iff v_1 = -v_2$ then the associated eigenvector is of the form $v_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and the corresponding solution is $\phi_1(t) = e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

the form $v_1\begin{pmatrix}1\\-1\end{pmatrix}$ and the corresponding solution is $\phi_1(t) = e \begin{pmatrix}1\\-1\end{pmatrix}$

$\boxed{\text{For } \lambda_2 = -1}$ solve $\begin{bmatrix}-2+1 & 1\\ 1 & -2+1\end{bmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Leftrightarrow$

$$\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Leftrightarrow \left.\begin{matrix}-v_1+v_2=0\\ v_1-v_2=0\end{matrix}\right\} \Rightarrow v_1 = v_2$$

the associated eigenvector is of the form $v_1\begin{pmatrix}1\\1\end{pmatrix}$ and the corresponding solution is $\phi_2(t) = e^{-t}\begin{pmatrix}1\\1\end{pmatrix}$

$\therefore$ the F Matrix for this system is $\boxed{\Phi(t) = \begin{pmatrix}e^{-3t} & e^{-t}\\ -e^{-3t} & e^{-t}\end{pmatrix}}$ and the general solution is $\boxed{\vec{Y}(t) = \Phi(t)\begin{pmatrix}c_1\\ c_2\end{pmatrix}}$